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Unitary representations of the q-oscillator algebra

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Abstract. We classify the unitary irreducible representations of the q-oscillator algebra $bb^+ - q^2b^+b = 1$ by the sign of the definite operator $K = bb^+ - b^+b$: $K > 0$ corresponds to Fock representations and $K < 0$ to non-Fock ones, $K = 0$ being a degenerate case. We present their link to representations of extended q-oscillator algebra $aa^+ - qa^+a = q^{-N}$ obtained earlier. We give examples of all representations. Besides standard ones, we describe a new non-Fock representation of the Macfarlane type.

1. Introduction

The Heisenberg oscillator algebra of annihilaton and creation operators plays a central role in the quantum physics and in the theory of representations of Lie groups. Similarly, the q-oscillator algebra already known to Heisenberg (as mentioned in [1]) and rediscovered by many other authors [2–9] is important for the construction of q-deformed Lie algebras. The q-oscillators appear basically either in the form

$$bb^+ - q^2b^+b = 1 \tag{1}$$

or

$$\begin{aligned} &aa^+ - qa^+a = q^{-N} \\ &[N, a] = -a \qquad [N, a^+] = a^+. \end{aligned} \tag{2}$$

While (1) is a deformation of the original oscillator algebra

$$b_0b_0^+ - b_0^+b_0 = 1 \tag{3}$$

Equation (2) is a deformation of the extended oscillator algebra

$$\begin{aligned} &a_0a_0^+ - a_0^+a_0 = 1 \\ &[N, a_0] = -a_0 \qquad [N, a_0^+] = a_0^+. \end{aligned} \tag{4}$$

We stress that (3) and (4) are different objects, although they are closely related. Putting

$$\begin{aligned} &N = M + \omega \qquad \omega \in \boldsymbol{R} \\ &a_0 = b_0 \qquad a_0^+ = b_0^+ \end{aligned} \tag{5}$$

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where ω is a real parameter and $M = b_0^+ b_0$, we obtain a particular realization of algebra (4) in terms of algebra (3). It is well known that all unitary irreducible representations of (4) are expressed in this way in terms of a unique (up to equivalence) unitary representation of (3) (see e.g. [11, 12]). The additional parameter ω is related to the central element

$$Z = N - a^+ a \tag{6}$$

of algebra (4) (which is identically equal to zero in the algebra (3)).

In this paper we extend these results to the unitary irreducible representations of the q-oscillator algebras (1) and (2). The representations of q-oscillator algebras were investigated in [13–16]. Reference [14] deals with the general algebraic properties of irreducible epresentations of q-oscillator algebras over an arbitrary field. The unitary irreducible representations of the q-oscillator algebra (2) were investigated in [15, 16].

In section 2 we classify unitary irreducible representations of the q-oscillator algebra (1), and we show that their relation to unitary irreducible representations of the algebra (2) is given by a direct generalization of (5) and (6). Section 3 contains particular realizations of all representations in question. Besides standard representations, we discuss a new non-Fock representation of the Macfarlane type (proposed originaly in the Fock case). Section 4 presents concluding remarks.

2. Classification theorem

The unitary irreducible representations of the q-oscillator algebra (1) are classified by the following

Theorem 1. Let b be a closed densely defined operator in a Hilbert space $\mathcal{H}$, let b^+ be the adjoint of b, and suppose that

$$bb^+ - q^2 b^+ b = 1 \qquad q > 0 \tag{7}$$

that is, (7) holds on a dense domain $D_{bb^+} = D_{b^+b}$. Then b^+b is self-adjoint, and the irreducible representations of (1) are:

(A) the Fock representation for any $q > 0$ with a non-degenerate spectrum of b^+b given by the formula

$$\lambda_k = \frac{1 - q^{2k}}{1 - q^2} =: [k] \qquad k = 0, 1, 2, \ldots \tag{8}$$

($[k] = k$ for $q = 1$). If $0 < q < 1$, the operators b and b^+ are bounded,

(B) the non-Fock representations for $0 < q < 1$ with the non-degenerate spectrum of b^+b given by

$$\lambda_k = \frac{1 + q^{2k+2\gamma}}{1 - q^2} =: \{k + \gamma\} \qquad k \in Z \tag{9}$$

and the representations are classified by $\gamma \in [0, 1)$, and

(C) the degenerate representation for $0 < q < 1$ with

$$b^+ b = bb^+ = (1 - q^2)^{-1} I. \tag{10}$$

To prove this theorem we define the self-adjoint operator

$$K = bb^+ - b^+ b. \tag{11}$$

From equation (7) it follows that K has the property

$$Kb = q^{-2} bK \qquad Kb^+ = q^2 b^+ K. \tag{12}$$

This equation gives

$$b^+Kb = q^{-2}b^+bK \qquad bKb^+ = q^2bb^+K. \tag{13}$$

Since b^+b and bb^+ are positive operators commuting with K, we see that K is definite on a subspace $\mathcal{H}$ spanned by vectors $(b^+)^m|k\rangle$, $b^m|k\rangle$, $m = 0, 1, 2, \ldots$, where $|k\rangle$ is some eigenstate of K. Thus the representations are classified by the sign of the commutator K.

The case $K = 0$ just corresponds to case (C), and (10) follows immediately.

If $K \neq 0$, we put $|K| = q^{2M}$. From (12) we obtain

$$[M, b] = -b, \ [M, b^+] = b^+. \tag{14}$$

The operator $\exp(2\pi iM)$ then commutes with b and b^+ and has in any irreducible representation the fixed value

$$\exp(2\pi iM) = \exp(2\pi i\gamma) \qquad \gamma \in [0, 1). \tag{15}$$

Consequently, M has a discrete spectrum containing points of the form $k+\gamma$, with k integer. Let $|k\rangle$ be a normalized eigenstate

$$M|k\rangle = (k+\gamma)|k\rangle. \tag{16}$$

Using (14), one can show that $b^+|k\rangle$ and $b|k\rangle$ are the eigenstates of M (provided that they are non-vanishing) to the eigenvalues $k+1+\gamma$ and $k-1+\gamma$, respectively.

(A) Let $K = q^{2M} > 0$. Then from (7) and (11) we obtain

$$b^+b = \frac{1-q^{2M}}{1-q^2} = [M]\,. \tag{17}$$

As $[k+\gamma] < 0$ for $k+\gamma < 0$, we see that in this case only $\gamma = 0$ is allowed. Then there exists a system of normalized eigenstates $|k\rangle$, $k = 0, 1, 2, \ldots$, such that

$$M|k\rangle = k|k\rangle \qquad k = 0, 1, 2, \ldots\,. \tag{18}$$

Equation (8) then follows directly. We note that

$$b^+|k\rangle = [k+1]^{1/2}|k+1\rangle \qquad b|k\rangle = [k]^{1/2}|k-1\rangle. \tag{19}$$

We see that all states $|k\rangle$, $k = 1, 2, \ldots$, can be obtained by the repeated action of b^+ on the vacuum state $|0\rangle$, satisfying $b|0\rangle$. We refer to this as 'the Fock case'.

(B) If $K = -q^{2M} < 0$, then

$$b^+b = \frac{1+q^{2M}}{1-q^2} = \{M\}. \tag{20}$$

As b^+b is positive, only $0 < q < 1$ is allowed. The normalized eigenstates $|k\rangle$, $k \in \boldsymbol{Z}$ of M then satisfy

$$M|k\rangle = (k+\gamma)|k\rangle \qquad k \in \boldsymbol{Z}. \tag{21}$$

Equation (9) follows immediately. In this case

$$b^+|k\rangle = \{k+\gamma+1\}^{1/2}|k+1\rangle \qquad b|k\rangle = \{k+\gamma\}^{1/2}|k-1\rangle. \tag{22}$$

Since now there is no vacuum state, we refer to this as to the non-Fock case. This completes the proof of theorem 1.

3. Representations

We describe two types of representations of q-oscillator algebra (1), that is, the Bargmann and the Macfarlane representations.

We briefly describe the Bargmann (holomorphic) represetation in the Fock case (see [2]). The space $\mathcal{H}$ is spanned by functions of the complex variable z:

$$\phi_k(z) = ([k]!)^{-1/2} z^k \qquad k = 0, 1, 2, \ldots \tag{23}$$

where $[k]! = [1][2]\ldots[k]$. They are eigenfunctions of the operator $M = z\partial_z$ to the eigenvalue k. The operators b and b^+ have the form

$$b^+ = z \qquad b = \frac{1}{z}[M]. \tag{24}$$

In $\mathcal{H}$ the scalar product is defined by $(\phi_n, \phi_m) = \delta_{n,m}$, and can be expressed in terms of the Jackson integral.

The functions (23) can be reinterpreted as coefficients of q-coherent states in the basis $|k\rangle$, $k = 0, 1, 2, \ldots$ (see [15-17]):

$$\langle k|z\rangle = \phi_k^*(y) \qquad k = 0, 1, 2, \ldots . \tag{25}$$

For $q > 1$, all $z \in C$ are admissible, whereas $|z| < (1-q^2)^{-1}$ for $0 < q < 1$.

The standard Macfarlane's q-oscillator representation [6] is defined by the operators

$$\begin{aligned} \tilde{b} &= \frac{1}{1-q^2}[\mathrm{e}^{-2\mathrm{i}s\varphi} - \mathrm{e}^{-\mathrm{i}s\varphi}\mathrm{e}^{\mathrm{i}s\partial}] \\ \tilde{b}^+ &= \frac{1}{1-q^2}[\mathrm{e}^{2\mathrm{i}s\varphi} - \mathrm{e}^{\mathrm{i}s\partial}\mathrm{e}^{\mathrm{i}s\varphi}] \end{aligned} \tag{26}$$

where $\partial = \partial_\varphi$ and $0 < q = \exp(-s^2) < 1$. Operators (26) formally act on a suitable subset of the functions defined on the interval $J = (-\frac{\pi}{2s}, +\frac{\pi}{2s})$ and belonging to the Hilbert space $\mathcal{L}_2(J, |h(\varphi)|^2\,\mathrm{d}\varphi)$, where

$$h(\varphi) = \prod_{m=0}^{\infty}[1 + \mathrm{e}^{-s^2(2m+1)+2\mathrm{i}s\varphi}]. \tag{27}$$

We shall use slightly different Macfarlane representations given by the operators

$$\begin{aligned} b &= \frac{1}{1-q^2}[\mathrm{e}^{-2\mathrm{i}s\varphi} \mp (\mathrm{e}^{-2\mathrm{i}s\varphi} + \mathrm{e}^{-s^2})\mathrm{e}^{\mathrm{i}s\partial}] \\ b^+ &= \frac{1}{1-q^2}[\mathrm{e}^{2\mathrm{i}s\varphi} \mp \mathrm{e}^{\mathrm{i}s\partial}(\mathrm{e}^{2\mathrm{i}s\varphi} + \mathrm{e}^{-s^2})]. \end{aligned} \tag{28}$$

They act in a suitable subspace $\mathcal{H}$ of functions from $\mathcal{L}_2(J, \mathrm{d}\varphi)$ specified below. Their commutator (11) is

$$K = \pm(1 + \mathrm{e}^{s^2-2\mathrm{i}s\varphi})(\mathrm{e}^{\mathrm{i}s\partial} \mp \mathrm{e}^{2\mathrm{i}s\partial})(1 + \mathrm{e}^{s^2+2\mathrm{i}s\varphi}). \tag{29}$$

The upper signs in (28), (29) correspond to the Fock representation in which the operators b and b^+ are related to $\tilde{b}$ and $\tilde{b}^+$ as

$$b = A(\varphi)\tilde{b}A^{-1}(\varphi) \qquad b^+ = A(\varphi)\tilde{b}^+A^{-1}(\varphi)$$

where $A(\varphi) = h(\varphi)\exp(\frac{1}{2}\mathrm{i}s\varphi)$.

In this case the operators b and b^+ acts in the space $\mathcal{H}$ spanned by the functions $\psi_k(\varphi) = \exp(2\mathrm{i}sk\varphi)$, $k = 0, 1, 2, \ldots$. The operator $\exp \mathrm{i}s\partial$ acts in $\mathcal{H}$ as

$$\mathrm{e}^{\mathrm{i}s\partial}\psi_k(\varphi) = \mathrm{e}^{-ks^2}\psi_k(\varphi). \tag{30}$$

Using this, one can show that the (unnormalized) vacuum state $\phi_0(\varphi)$ satisfying

$$K\phi_0(\varphi) = \phi_0(\varphi) \tag{31}$$

is

$$\phi_0(\varphi) = h(\varphi) \tag{32}$$

with $h(\varphi)$ given in (27). This follows from the relation

$$h(\varphi + \mathrm{i}s) = (1 + \mathrm{e}^{-s^2+2\mathrm{i}s\varphi})^{-1} h(\varphi). \tag{33}$$

The eigenstates $\phi_k(\varphi)$, $k = 1, 2, \ldots$, (with the same norm as $\phi_k(\varphi)$) are given according to (19) as

$$\phi_k(\varphi) = ([k]!)^{-1/2}(b^+)^k \phi_0(\varphi) \tag{34}$$

where b^+ is given in (28) with the upper signs.

The lower signs in (28), (29) correspond to a new non-Fock representation in the subspace $\mathcal{H}$ of $\mathcal{L}_2(J, d\varphi)$ spanned by the functions $\psi_k(\varphi) = \exp[2\mathrm{i}s(k+\gamma)\varphi]$, $k \in \mathbb{Z}$, where $\gamma \in [0, 1)$ is fixed. In this space the operator $\exp(\mathrm{i}s\partial)$ acts as

$$\mathrm{e}^{\mathrm{i}s\partial}\psi_k(\varphi) = \mathrm{e}^{-2(k+\gamma)s^2}\psi_k(\varphi) \qquad k \in \mathbb{Z}. \tag{35}$$

Obviously, the operator K (given in (29) with lower signs) is negative, so that we are really dealing with the non-Fock case.

The (unnormalized) eigenfunction $\phi_0^\gamma(\varphi)$ satisfying

$$-K\ \phi_0^\gamma(\gamma) = \mathrm{e}^{-2\gamma s^2}\phi_0^\gamma(\varphi) \tag{36}$$

is given by the formula

$$\phi_0^\gamma(\varphi) = \frac{1}{k(\varphi)}\sum_{m=0}^{\infty} a_m \mathrm{e}^{2\mathrm{i}s(m+\gamma)\varphi} \tag{37}$$

where

$$a_m = \mathrm{e}^{-m^2 s^2}\prod_{n=1}^{m}\frac{1+\mathrm{e}^{-2(n+\gamma)s^2}}{1-\mathrm{e}^{-2ns^2}} \tag{38}$$

$$k(\varphi) = \prod_{n=1}^{\infty}[1 + \mathrm{e}^{-(2n-1)s^2-2\mathrm{i}s\varphi}]. \tag{39}$$

Equation (36) follows from the relations

$$a_{m-1} = \mathrm{e}^{(2m-1)s^2}\frac{1-\mathrm{e}^{-2ms^2}}{1+\mathrm{e}^{-2(m+\gamma)s^2}}\, a_m \tag{40}$$

$$k(\varphi + \mathrm{i}s) = (1 + \mathrm{e}^{s^2-2\mathrm{i}s\varphi})k(\varphi) \tag{41}$$

and the formula (29) for K with lower signs.

The eigenstates $\phi_{\pm k}^\gamma(\varphi)$, $k = 1, 2, \ldots$, (with the same norm as $\phi_0^\gamma(\varphi)$) are given according to (22) as

$$\begin{aligned}\phi_{-k}^\gamma &= (\gamma\{\gamma + k - 1\})^{-1/2} b^k \phi_0^\gamma(\varphi)\\ \phi_k^\gamma &= (\{\gamma + k\}!)^{-1/2}\ (b^+)^k \phi_0^\gamma(\varphi)\end{aligned} \tag{42}$$

where $\{\gamma + k\}! = \{\gamma + 1\}\ldots\{\gamma + k\}$, and b and b^+ are defined in (28) with lower signs. We stress that the parameter γ introduced in (35) is identical to the one introduced in (15).

4. Concluding Remarks

We shall first compare our results to the ones found in [15] for the extended q-oscillator algebra. Putting

$$\begin{aligned} N &= M + \omega \qquad & n\omega &\in \boldsymbol{R} \\ a &= q^{-N/2} b \qquad & a^{+} &= b^{+} q^{-N/2} \end{aligned} \tag{43}$$

we obtain from the representations listed in theorem 1 all classes of unitary irreducible representations of the extended q-oscillator algebra (2) obtained in [15]. The additional parameter ω is related to the non-trivial central element of the algebra (2) found in [10],

$$Z = [N] - q^{-N+1} a^{+} a \tag{44}$$

(the analogous quantity in the algebra (1) is identically equal to zero).

The most important applications of q-oscillators are related to the construction of representations of q-deformed Lie algebras. In many cases the applications are based on the use of the Hayashi algebra [9] (a kind of the q-deformed Weyl algebra) defined by equations (2) and by the relation

$$a^{+} a = \frac{q^{N} - q^{-N}}{q - q^{-1}}. \tag{45}$$

We note, however, that this is just the relation valid in in unitary Fock representation of algebra (2) in the particular case of $\omega = 0$.

The unitary irreducible representations of the Hayashi algebra are uniquely related to the unitary irreducible Fock representations of algebra (1). They were intensely used for the construction of some representations of q-deformed Lie algebras [5–9]. The non-Fock representations of algebra (1) were used in [15] for the construction of representations of the $su_q(2)$ algebra. It would be interesting to extend such constructions with non-Fock oscillators to other q-deformed Lie algebras, e.g. along the lines performed in [9] with the Fock q-oscillators.

Moreover, there are many important constructions of representations of q-deformed Lie algebras (see e.g. [18, 19]) but the questions concerning their unitarity usualy remain open. We hope that the use of various Fock and non-Fock q-oscillators may shed light on these problems.

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